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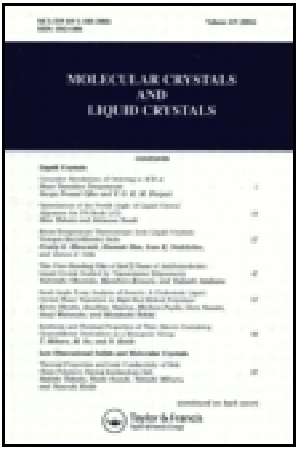

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Waveguiding in a Nematic Hybrid Slab

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Waveguiding in a Nematic Hybrid Slab

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We develop an analytical approach to study the propagation of optical fields through a nematic slab model with hybrid boundary conditions studied numerically by Lin *et al.* [1]. For a low intensity beam we first derive the eikonal equation and from it we then calculate the ray trajectories in the limit of geometrical optics. We show that there may exist caustics within the slab. Then we consider the WKB limit and calculate the field transverse magnetic modes, their number and their cutoff frequencies. We show that for both limits the agreement between the analytical and numerical results for the propagation constants is excellent, when caustics are not present; however, their presence introduces larger differences in the values of the calculated field amplitudes but the comparison is still quite good. We show that these discrepancies have their origin in the fact that the chosen parameter values in the exact numerical calculations, do not lie within the limits of validity of our WKB approximation. Thus, a more precise comparison between these approaches requires different sets of values of the relevant parameters. Finally, we discuss the scope and limitations of our method.

Keywords: Nematic slab; wave guide; WKB; geometrical optics

PACS Nos.: 42.65.Jx; 61.30.Gd; 64.70.Md; 78.20.Jq

1. INTRODUCTION

In a previous work [2], we have compared our analytical but approximated description of the propagation of optical fields in a cylindrical liquid crystal waveguide [3], with earlier numerical calculations performed by Lin and Palffy-Muhoray in a series of interesting works [4, 5]. We found that for

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nematic cores with negative birefringence, our analytically calculated values of the propagation constant are in excellent agreement with the above mentioned numerical results, but that this agreement is rather poor for the optical fields distributions. The basic purpose of this work is to extend this comparison for a simpler planar geometry and to show that even though the chosen parameter values in the numerical approach are, strictly speaking, out of the limit of the approximation we use, the agreement is good enough and better than the one obtained for the cylindrical case [2]. To this end we study the same model of a nematic slab with strong hybrid boundary conditions introduced in [1] and describe analytically the propagation of optical fields along the slab.

The paper is organized as follows. In Section 2, we calculate the ray trajectories within the slab for a low intensity beam in the limit of geometrical optics (LGO). Then in Section 3 we give the expressions for the TM modes to the next successive order approximation, namely, in the WKB limit. An expression for the cutoff frequency and for the number of TM modes is obtained in Section 4 and finally, in Section 5, we compare our analytical results with those in Ref. [1]. We show that they are in reasonably good agreement and we discuss the limitations and advantages of our method.

2. RAY TRAJECTORIES

We consider a nematic crystal layer of thickness l measured along the x axis and contained between two parallel isotropic dielectric media, as shown in Figure 1. The transverse dimensions along the z and y directions are large compared to l, but the cell has a finite volume V. We consider an initially hybrid configuration of the director field where it is parallel to the boundary at $x = 0$ and perpendicular to it on $x = l$. In the absence of external optical fields, the nematic will retain this initial orientation $\hat{n}(\theta)$. However, if the liquid crystal is excited by an obliquely applied laser beam in the $x-z$ plane, the orientation of the director inside the cell will change with position and time. In the absence of backflows, it is to be expected that the polarization of the beam will remain in the plane of incidence and that the reorientation of $\hat{n}$ will also take place in the $x-z$ plane. Thus we have

$$\hat{n} = (\sin\theta(x,t), 0, \cos\theta(x,t)), \tag{1}$$

where θ is the reorientation angle defined with respect to the z axis.

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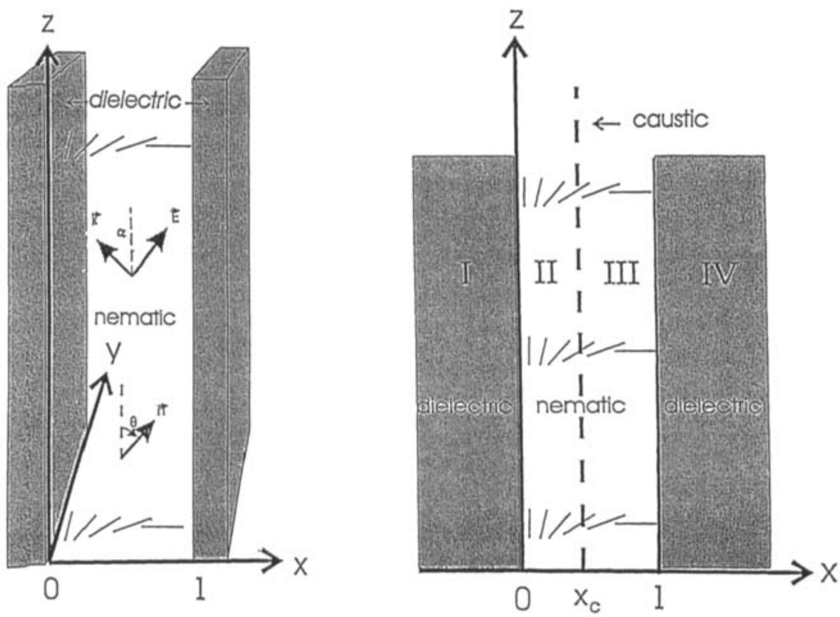

FIGURE 1 Schematics of a hybrid slab waveguide.

In Ref. [1], Lin *et al.* studied the propagation of waves within a planar nematic cell in terms of the complete representation provided by the corresponding TM modes $E_x(x,\ k_0)$, $E_z(x,\ k_0)$ and $H_y(x,\ k_0)$, where $k_0 \equiv (\omega/c)$ is the free space wave number, being ω the frequency of the field and c the speed of light in vacuum. These modes are the only ones that couple to the reorientation of the director. Using Maxwell's equations without sources to describe the optical field within the liquid crystal, and assuming the modes to have a time and z coordinate dependence given by $e^{i(\omega t-\beta z)}$, a set of equations for the amplitudes of the TM modes may be derived. When rewritten in terms of the dimensionless variable $\zeta \equiv (x/l)$ and in gaussian units these equations read

$$\varepsilon_{xx}\frac{d^2H_y}{d\zeta^2}+\left[-2ik_0lp\varepsilon_{xz}+\frac{d\varepsilon_{xx}}{d\zeta}\right]\frac{dH_y}{d\zeta}+k_0l\left[k_0l(\varepsilon_{\parallel}\varepsilon_{\perp}-p^2\varepsilon_{zz})-ip\frac{d\varepsilon_{xz}}{d\zeta}\right]H_y=0, \tag{2}$$

$$E_z=-\frac{1}{\varepsilon_{\parallel}\varepsilon_{\perp}}\left\{\varepsilon_{xz}pH_y+\frac{i}{k_0l}\varepsilon_{xx}\frac{dH}{d\zeta}\right\}, \tag{3}$$

$$E_x = \frac{1}{\varepsilon_\| \varepsilon_\perp} \left\{ p\varepsilon_{zz} H_y + \frac{i}{k_0 l} \varepsilon_{xz} \frac{dH}{d\zeta} \right\}. \tag{4}$$

Here $p \equiv (\beta/k_0)$, being β the propagation constant and where $\varepsilon_\|$ and $\varepsilon_\perp$ denote, respectively, the dielectric constants parallel and perpendicular to the long axis of the molecules. $\varepsilon_a = \varepsilon_\| - \varepsilon_\perp$ is the dielectric anisotropy of the nematic and the components of the dielectric tensor $\varepsilon_{ij} = \varepsilon_\perp \delta_{ij} + \varepsilon_a n_i n_j$, are explicit functions of θ, that is, $\varepsilon_{zz} = \varepsilon_\perp + \varepsilon_a \cos^2\theta(\zeta)$, $\varepsilon_{xx} = \varepsilon_\perp + \varepsilon_a \sin^2\theta(\zeta)$ and $\varepsilon_{xz} = \varepsilon_a \sin\theta(\zeta)\cos\theta(\zeta)$. Since in what follows we shall utilize some expressions derived in earlier works [3, 6], it should be mentioned that here we use a different coordinate system where the variables x and z are interchanged and the time and z dependence of the modes is given by $e^{i(\beta z - \omega t)}$. Hence, when we invoke an expression from Refs. [3, 6], the variables x and z should be interchanged, $p = (c\beta/\omega)$ remains the same and $k_0 l = (\omega l/c)$ changes sign.

The stationary orientational configurations are determined by minimizing the free energy functional of the model, Eq. (4) in [3]. In particular, the final stationary orientational state θ after the reorientation has occured, is defined by the corresponding Euler-Lagrange equation, which if written in dimensionless form, reads

$$\frac{d^2\theta}{d\zeta^2} + q\left[\sin 2\theta(|\overline{E}_x|^2 - |\overline{E}_z|^2) + (\overline{E}_x \overline{E}_z^* + \overline{E}_z \overline{E}_x^*)\cos 2\theta\right] = 0. \tag{5}$$

Here we have introduced the dimensionless fields $\overline{E}_x = (E_x/E_0)$ and $\overline{E}_z = (E_z/E_0)$ where E_0 is given in terms of the intensity $I = (E_0^2/8\pi)$ of the incident field and $*$ indicates complex conjugate. The parameter $q \equiv \varepsilon_a(E_0^2 l^2/8\pi K)$, where K is the elastic constant in the equal constant approximation, is proportional to the ratio between the electric energy of the beam to the elastic energy of the nematic and, therefore, measures the strength of the coupling between the optical field and the orientational configuration of the nematic.

Now, since the concept of ray trajectory is defined only in the LGO and the TM modes will be required in the WKB limit later on, we recall the definition of these limits in terms of the values of the parameter $k_0 l$ [7]. The LGO of Eqs. (2–4) corresponds to keeping terms such that $k_0 l \gg 1$, while the WKB limit implied keeping terms for which $k_0 l > 1$. Therefore, in order to calculate the ray trajectories for a low intensity incident beam, it is necessary to solve Eqs. (2–4), respectively, up to terms of orders zero and one in the parameter $(1/k_0 l)$. In previous works [3, 6] we have already

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obtained the general expressions for the TM modes in these limits. Actually, interchanging the subscripts x and z in the corresponding dielectric tensor components, from Eq. (44) of [3] we obtain for the ray trajectory

$$\gamma = \chi - \int_0^{\zeta} d\eta \left[\frac{\varepsilon_{xz}}{\varepsilon_{xx}} + \frac{p}{\varepsilon_{xx}} \sqrt{\frac{\varepsilon_{\perp}\varepsilon_{\|}}{\varepsilon_{xx} - p^2}} \right], \tag{6}$$

where $\chi \equiv (z/l)$ and γ is the invariant generalized coordinate conjugated to p. This identification is established on the basis of the transformations used in the Hamilton-Jacobi theory, according to which all the new variables are invariant in time; that is, they are initial conditions or constants of motion. The propagation constant p may be expressed in terms of the propagation angle α defined in Figure 1 by evaluating $(d\chi/d\zeta) = -\cot\alpha$ at $\zeta \equiv (x/l) = 1$ from Eq. (6). This leads to

$$p = \sqrt{\varepsilon_{\|}} \frac{\cot\alpha}{\sqrt{\cot^2\alpha + \frac{\varepsilon_{\perp}}{\varepsilon_{\|}}}}. \tag{7}$$

However, to calculate the trajectory explicitly, it is necessary to determine first $\varepsilon_{ij}(\theta)$. In this work we shall only consider the case in which the dynamics of the field is completely decoupled to the orientational dynamics, *i.e.*, $q = 0$. Then, from Eq. (5) and the hybrid boundary conditions $\theta(x = 0) = 0$ and $\theta(x = 1) = \pi/2$, the stationary configurational state turns out to be

$$\theta = \frac{\pi}{2}\zeta. \tag{8}$$

With this result, the tensor components in Eq. (6) can be determined as well defined functions of x. It is easy to show that when the parameter p is in the interval $\varepsilon_c < p^2 < \varepsilon_{\perp}$, which we define as the *weak regime* (*WR*), the ray trajectory exists for *all* points within the cell ε_c is the dielectric constant of the claddings. On the other hand, in the *strong regime* (*SR*), that is, when $\varepsilon_{\perp} < p^2 < \varepsilon_{\|}$, there are values of ζ for which the quantity $\sqrt{\varepsilon_{xx} - p^2}$ in the denominator of Eq. (6) vanishes. These values define the position $\zeta_c \equiv (x_c/l)$ of the caustics or asymptotes of the ray trajectories. It can be easily verified that in this case there is only one caustic within the slab located at

$$\zeta_c = \frac{2}{\pi} \arccos \sqrt{\frac{\varepsilon_{\|} - p^2}{\varepsilon_a}}. \tag{9}$$

This means that the ray is constrained to be localized on the right hand side of the slab.

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In Figures 2 we plot two ray trajectories for the incidence angles $\alpha_I = 25°$ and $\alpha_I = 60°$, respectively, for the nematic phase of cyanobiphenil. Figure 2a shows the existence of a caustic which does not allow the ray to reach the left hand side of the slab, while Figure 2b shows that there is no caustic and therefore the ray may exists in the whole slab.

3. TM MODES

Let us now study the TM modes separately in the *WR* and *SR*. For the former case we may use Eq. (10) of [6], which gives the general solution of Eq. (2) valid in the WKB limit when there are no caustics in the nematic slab. It is given by

$$H_y(\zeta, k_0) = \frac{1}{\sqrt[4]{\varepsilon_\perp \varepsilon_\| (\varepsilon_{xx} - p^2)}} e^{i\phi(\zeta, k_0)} \left[C \cos\left(k_0 l \int_0^\zeta d\eta f(\eta, k_0) \right) + D \sin\left(k_0 l \int_0^\zeta d\eta f(\eta, k_0) \right) \right], \tag{10}$$

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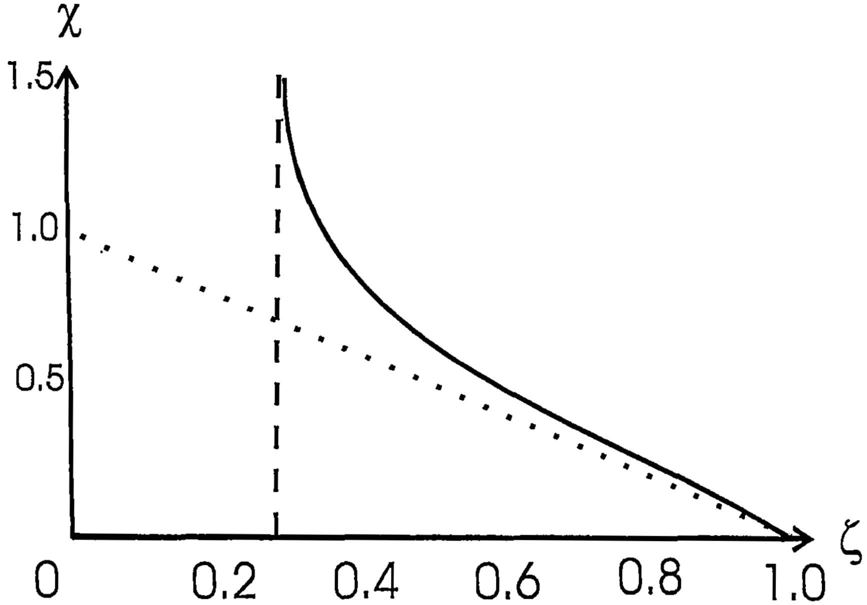

FIGURE 2a Ray trajectories for $\alpha_I = 25°$ for cyanobiphenil (—); (- - -) denotes the isotropic case $\varepsilon_a = 0$.

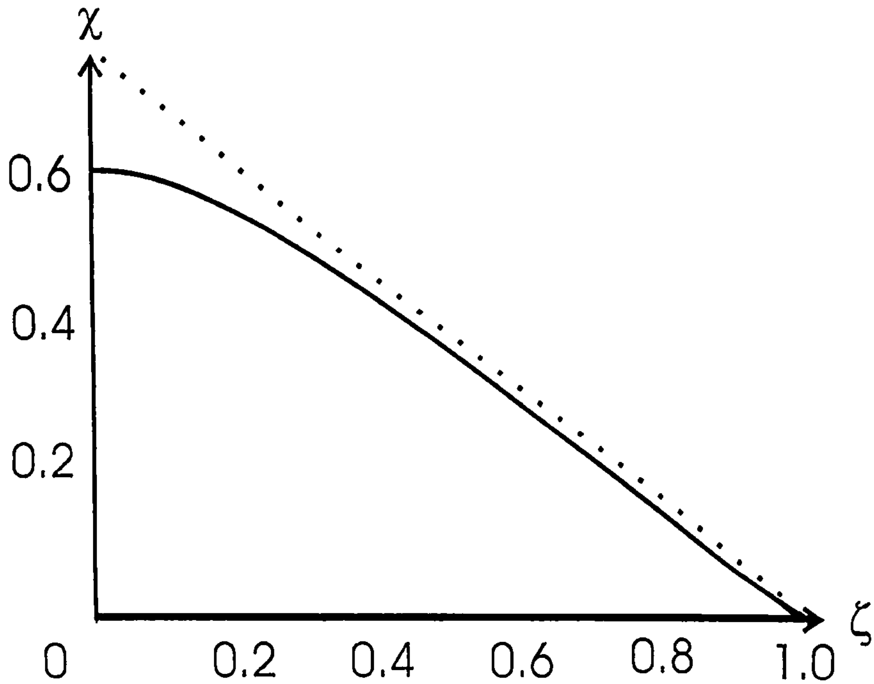

FIGURE 2b The same as in Figure 2a for $\alpha_l = 60°$.

where C and D are two arbitrary constants to be determined by the boundary conditions. The functions $\phi(\zeta, k_0)$ and $f(\eta, k_0)$ are defined, respectively, by

$$\phi(\zeta, k_0) \equiv pk_0 l \int_0^\zeta d\eta \frac{\varepsilon_{xz}}{\varepsilon_{xx}} = \frac{pk_0 l}{\pi} \ln\left(\frac{\varepsilon_{xx}}{\varepsilon_\perp}\right), \tag{11}$$

$$f(\eta, k_0) \equiv \frac{\sqrt{\varepsilon_\perp \varepsilon_\parallel (\varepsilon_{xx} - p^2)}}{\varepsilon_{xx}}. \tag{12}$$

On the other hand, $H_y(\zeta, k_0)$ in the isotropic dielectric claddings is governed by the following equation [8].

$$\frac{d^2 H_y}{d\zeta^2} + (k_0 l)^2 (\varepsilon_c - p^2) H_y = 0. \tag{13}$$

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Solutions of Eq. (13) which vanish at infinity for the left, H_y^L, and right, H_y^R, dielectric claddings are given by

$$H_y^L = Fe^{k_0 l\sqrt{p^2-\varepsilon_c}\zeta}, \tag{14}$$

$$H_y^R = Ge^{-k_0 l\sqrt{p^2-\varepsilon_c}\zeta}, \tag{15}$$

where G and F are also undetermined constants. To find the four undetermined constants C, D, G and F, it is necessary to impose the boundary conditions over H_y and its derivate at both slab boundaries. These conditions may be written as [9]

$$H_y^L|_{\zeta=0} = H_y|_{\zeta=0}, \tag{16}$$

$$\frac{1}{\varepsilon_c}\frac{dH_y^L}{d\zeta}\bigg|_{\zeta=0} = \frac{1}{\varepsilon_\parallel}\frac{dH_y}{d\zeta}\bigg|_{\zeta=0}, \tag{17}$$

$$H_y^R|_{\zeta=1} = H_y|_{\zeta=1}, \tag{18}$$

$$\frac{1}{\varepsilon_c}\frac{dH_y^R}{d\zeta}\bigg|_{\zeta=1} = \frac{1}{\varepsilon_\perp}\frac{dH_y}{d\zeta}\bigg|_{\zeta=1}. \tag{19}$$

If we now insert Eqs. (10), (14) and (15) into these boundary conditions, place the origin of the reference system at the left boundary of the slab and solve the resulting system of equations, we find expressions for the TM modes in terms of p or α. However, not all the values of α generate a propagation TM mode. The discret values that do so will be denoted by p_n, where the subindex n identifies the corresponding TM mode. For WR the expressions for $H_y(z, k_0)$ for all values of ζ are

$$H_{yn}^L = \frac{D}{\sqrt{\varepsilon_\parallel\varepsilon_\perp}}\frac{\varepsilon_c}{\sqrt{p_n^2-\varepsilon_c}}\sqrt[4]{\frac{\varepsilon_\perp - p_n^2}{\varepsilon_\parallel\varepsilon_\perp}}e^{k_0 l\sqrt{p_n^2-\varepsilon_c}\zeta}, \tag{20}$$

$$\begin{aligned} H_{yn} = &\frac{D}{\sqrt[4]{\varepsilon_\perp\varepsilon_\parallel(\varepsilon_{xx}-p_n^2)}}e^{i\phi(\zeta,k_0)}\left[\sin\left(k_0 l\int_0^\zeta d\eta f(\eta,k_0)\right)\right.\\ &\left.+\varepsilon_c\sqrt{\frac{\varepsilon_\perp - p^2}{\varepsilon_\perp\varepsilon_\parallel(p_n^2-\varepsilon_c)}}\cos\left(k_0 l\int_0^\zeta d\eta f(\eta,k_0)\right)\right], \end{aligned} \tag{21}$$

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$$H_{yn}^{R} = \frac{D}{\sqrt[4]{\varepsilon_{\|}\varepsilon_{\perp}(\varepsilon_{\|} - p_n^2)}} e^{\phi_c + \sqrt{p_n^2 - \varepsilon_c}\, k_0 l(1-\zeta)} \times \left[\sin\phi_c + \varepsilon_c \sqrt{\frac{\varepsilon_{\perp} - p_n^2}{\varepsilon_{\|}\varepsilon_{\perp}(p_n^2 - \varepsilon_c)}} \cos\phi_c\right], \tag{22}$$

where $\phi_c \equiv k_0 l \int_0^1 d\eta f(\eta, k_0)$. The remaining TM mode components $E_x(x, k_0)$ and $E_z(x, k_0)$ are obtained from Eqs. (4) and (3), respectively. The remaining constant D can be determined by the condition $\int_{-\infty}^{\infty} |H_{yn}|^2 (\zeta) d\zeta = 1$. On the other hand, from Eqs. (10), (14) and (15), it also follows that the allowed values of p_n are given by the solution of the trascendental equation

$$\frac{\tan\phi_c - \frac{1}{\varepsilon_c}\sqrt{\frac{\varepsilon_{\|}\varepsilon_{\perp}(p_n^2 - \varepsilon_c)}{\varepsilon_{\perp} - p_n^2}}}{1 + \tan\phi_c \frac{1}{\varepsilon_c}\sqrt{\frac{\varepsilon_{\|}\varepsilon_{\perp}(p_n^2 - \varepsilon_c)}{\varepsilon_{\perp} - p_n^2}}} = \frac{1}{\varepsilon_c}\sqrt{\frac{\varepsilon_{\|}\varepsilon_{\perp}(p_n^2 - \varepsilon_c)}{\varepsilon_{\|} - p_n^2}}. \tag{23}$$

Let us now derive an expression for the cutoff frequency, ω_{cn}^{WR}, for the TM mode of order n and for the maximum number of modes in *WR*. This is accomplished by setting $p_n^2 = \varepsilon_c$ in Eq. (23), which is the minimum value of p_n in order to have a propagating TM mode in the slab. This condition leads to the following expression for ω_{cn}^{WR}

$$\omega_{cn}^{WR} = \frac{nc\pi}{l} \Delta^{WR}, \tag{24}$$

with $n = 1, 2, \ldots$ and

$$\frac{1}{\Delta^{WR}} = \frac{2}{\pi}\sqrt{\frac{\varepsilon_{\|}\varepsilon_{\perp}}{\varepsilon_{\|} - \varepsilon_c}} \left[K\left(\frac{\varepsilon_a}{\varepsilon_{\|} - \varepsilon_c}\right) - \frac{\varepsilon_c}{\varepsilon_{\|}} \Pi\left(\frac{\varepsilon_a}{\varepsilon_{\|}}, \frac{\varepsilon_a}{\varepsilon_{\|} - \varepsilon_c}\right)\right], \tag{25}$$

and K, Π are the complete elliptic functions of first and third kinds [11], respectively. From the above expression it follows that the maximum number of propagating modes for a given frequency ω is determined by comparing ω with ω_{cm}. That is, if $\omega_{cm} < \omega < \omega_{c(m+1)}$, the maximum number of modes in the weak regime is m. Thus from this inequality and Eq. (24) we have that

$$m^{WR} = \frac{k_0 l}{\Delta^{WR} \pi}. \tag{26}$$

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Let us now consider SR, for which there exist a caustic located at ζ_c. In the presence of a caustic, Eq. (10) is no longer valid and in order to determine the mode H_y outside the caustic one should use instead the so called *connection formulas.* These relations are well known from the semiclassical approximation in quantum mechanics and are valid in the neighborhood of a turning point [10]

$$\frac{1}{\sqrt{\Gamma(\zeta)}}\exp\left[\int_{\zeta}^{\zeta_c}\Gamma(\eta)\,d\eta\right]\longrightarrow -\frac{1}{\sqrt{f(\zeta)}}\sin\left(\int_{\zeta_c}^{\zeta}f(\eta)\,d\eta-\frac{\pi}{4}\right), \tag{27}$$

$$\frac{1}{\sqrt{\Gamma(\zeta)}}\exp\left[-\int_{\zeta}^{\zeta_c}\Gamma(\eta)\,d\eta\right]\longrightarrow \frac{2}{\sqrt{f(\zeta)}}\cos\left(\int_{\zeta_c}^{\zeta}f(\eta)\,d\eta-\frac{\pi}{4}\right), \tag{28}$$

where we have defined $\Gamma(\eta)\equiv if(\eta)$. These expressions indicate that a solution of the form on the left hand side of the arrow, should be replaced by the corresponding one on the right hand side. The use of the above formulas with the boundary conditions Eqs. (16–19) lead to the following expressions for the TM modes for SR in all regions of space:

$$H_{yn}^{I}=\frac{A}{\sqrt[4]{\varepsilon_{\|}\,\varepsilon_{\perp}\,(p_n^2-\varepsilon_{\perp})}}\,\frac{2\Upsilon}{\Upsilon-1}\exp\left[k_0 l\sqrt{p_n^2-\varepsilon_c}\;\zeta\right], \tag{29}$$

$$\begin{aligned} H_{yn}^{II}=&\frac{A}{\sqrt[4]{\varepsilon_{\perp}\,\varepsilon_{\|}\,(p_n^2-\varepsilon_{xx})}}\,e^{i\phi(\zeta,k_0)}\\ &\times\left[\frac{\Upsilon+1}{\Upsilon-1}\,e^{k_0 l\int_0^{\zeta}d\eta\Gamma(\eta,k_0)}+e^{-k_0 l\int_0^{\zeta}d\eta\Gamma(\eta,k_0)}\right], \end{aligned} \tag{30}$$

$$\begin{aligned} H_{yn}^{III}=&\frac{A}{\sqrt[4]{\varepsilon_{\perp}\,\varepsilon_{\|}(\varepsilon_{xx}-p_n^2)}}\,e^{i\phi(\zeta,k_0)}\left[2\frac{\Upsilon+1}{\Upsilon-1}\,e^{\phi_1}\cos\left(k_0 l\int_{\zeta_c}^{\zeta}d\eta f(\eta,k_0)-\frac{\pi}{4}\right)\right.\\ &\left.-e^{-\phi_1}\sin\left(k_0 l\int_{\zeta_c}^{\zeta}d\eta f(\eta,k_0)-\frac{\pi}{4}\right)\right], \end{aligned} \tag{31}$$

$$\begin{aligned} H_{yn}^{IV}=&A\,\frac{e^{k_0 l\sqrt{p_n^2-\varepsilon_c}\,(1-\zeta)+i\phi_c}}{\sqrt[4]{\varepsilon_{\|}\,\varepsilon_{\perp}\,(\varepsilon_{\|}-p_n^2)}}\\ &\times\left[2\frac{\Upsilon+1}{\Upsilon-1}\,e^{\phi_1}\cos\left(\phi_2-\frac{\pi}{4}\right)-e^{-\phi_1}\sin\left(\phi_2-\frac{\pi}{4}\right)\right]. \end{aligned} \tag{32}$$

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Here we have introduced the abbreviations Υ and Λ defined by

$$\Upsilon \equiv \frac{\varepsilon_c \sqrt{p_n^2 - \varepsilon_\perp}}{\sqrt{\varepsilon_\perp \, \varepsilon_\parallel \, (p_n^2 - \varepsilon_c)}}, \tag{33}$$

$$\Lambda \equiv \frac{\varepsilon_c \sqrt{\varepsilon_\parallel - p_n^2}}{\sqrt{\varepsilon_\perp \, \varepsilon_\parallel \, (p_n^2 - \varepsilon_c)}}. \tag{34}$$

Again, the TM mode components $E_x(x, k_0)$ and $E_z(x, k_0)$ are obtained from Eqs. (4) and (3), respectively and the remaining constant A can be determined by the condition $\int_{-\infty}^{\infty} |H_{yn}|^2 (\zeta)\, d\zeta = 1$.

On the other hand, from the connection formulas and Eqs. (16–19), it also follows that for SR the allowed values of p_n are given by the solution of the trascendental equation

$$\frac{1 + \Upsilon}{\Upsilon - 1} = \frac{e^{-2\phi_1}}{2} \, \frac{\tan(\phi_2 - \frac{\pi}{4}) + \Lambda}{1 - \Lambda \tan(\phi_2 - \frac{\pi}{4})}, \tag{35}$$

with $\phi_1 \equiv k_0 l \int_0^{\zeta_c} d\eta \Gamma(\eta, k_0)$ and $\phi_2 \equiv k_0 l \int_{\zeta_c}^1 d\eta f(\eta, k_0)$.

Now, the expression for the cutoff frequency ω_{cn} for the TM mode of order n and for the maximum number of modes in the strong regime is obtained by setting $p_n^2 = \varepsilon_\perp$ in Eq. (35), which is the minimum value of p_n in order to have a propagating TM mode in this regime (SR). Since with this condition Υ, ζ_c and ϕ_1 vanish, we arrive at the following expression for ω_{cn}

$$\omega_{cn}^{SR} = \frac{c\pi}{l} \Delta^{SR} \left[n - \frac{3}{4} + \frac{1}{\pi} \arctan\left(\frac{2 + \Lambda^0}{-1 + 2\Lambda^0} \right) \right], \tag{36}$$

with $n = 1, 2, \ldots$ and

$$\frac{1}{\Delta^{SR}} \equiv \frac{2\sqrt{\varepsilon_\perp}}{\pi} \tanh^{-1} \sqrt{\frac{\varepsilon_a}{\varepsilon_\parallel}}, \tag{37}$$

$$\Lambda^0 = \frac{\varepsilon_c \sqrt{\varepsilon_a}}{\sqrt{\varepsilon_\parallel \, \varepsilon_\perp \, (\varepsilon_\perp - \varepsilon_c)}}. \tag{38}$$

Here n has been chosen in such a way that for the already given material parameters, $n = 1$ is the smallest integer for which $\omega_{cn}^{SR} > 0$. Following the

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same reasoning as was done for the weak regime, Eq. (36) yields that the maximum number of propagating modes for a given frequency ω is given by

$$m^{SR} = \frac{k_0 l}{\Delta^{SR}\pi}. \tag{39}$$

4. DISCUSSION

Since Lin *et al.* [1] use the material parameters of the nematic phase of cyano biphenil, $\varepsilon_{\perp}=2.25$ and $\varepsilon_a=0.6399$, [5], our analysis is carried out for the following parameter values: $l\sqrt{\varepsilon_{\perp}}/\lambda = 1.5$, $\varepsilon_{\perp}/\varepsilon_c = 1.03$ and $\varepsilon_{\parallel}/\varepsilon_{\perp} = 1.2844$, where λ is the wavelength in free space. It then follows that $k_0 l = 2\pi l/\lambda = 6.28$. Note that according to the definition given in Section 2, this value barely corresponds to the WKB limit. It is straightforward to show that for this value of the parameter, there are only two TM modes in the cell. Actually, inserting $k_0 l = 6.28$ into Eqs. (24), (25), (26) and (36), (37), (39), respectively, it follows that $m^{WR} = 1$ and $m^{SR} = 1$. Thus, there are only two modes within the cell and one in each regime, *WR* and *SR*.

We now proceed to compare the predictions of our approach with those of Ref. [1] for each regime. Let us first calculate the magnitude and the phase of this TM mode in the *WR*. However, to do this it is necessary to solve first Eq. (23) for the same parameter values used in the numerical calculations. By plotting in Figure 3 the left and right hand sides of this equation as functions of the propagation constant p, its solutions are determined graphically. Note that the plot shows that there exist only one intersection at $p_1^{WR} = 1.48$. Substituting this value into Eqs. (20–22), we calculate the magnitude of the TM mode $|\overline{H}_{y02}|$. This yields the curve plotted in Figure 4a with $\overline{H}_y(\zeta) \equiv H_y(\zeta)/\int_{-\infty}^{\infty} |H_y|^2(\zeta)\,d\zeta$. If we compare the position where $|\overline{H}_{y02}|$ vanishes with the corresponding one obtained by Lin *et al.* [1], we find a difference of 0.4% between both values. On the other hand, the phase $\phi_{02}^{WR}(\zeta, k_0)$ of $|\overline{H}_{y02}|$, which is given by Eq. (11), turns out to be very close to the one obtained in Ref. [1] as shown in Figure 4b. Actually, although there is a 2% difference in their maximum positive values, this result confirms that there is indeed a good agreement for the weak regime between our analytical calculation and the numerical ones performed by Lin *et al.* [1].

Let us now calculate the same quantities for the *SR*. By following a similar procedure, we first determine the allowed values of the propagation constant in this case. Inserting the value $k_0 l = 6.28$ into (35) and solving graphically, we find that $p_1^{SR} = 1.35$, as shown in Figure 5. Substitution of this value into Eqs. (31) and (30) yields the magnitude of $|\overline{H}_{y01}|$, which is

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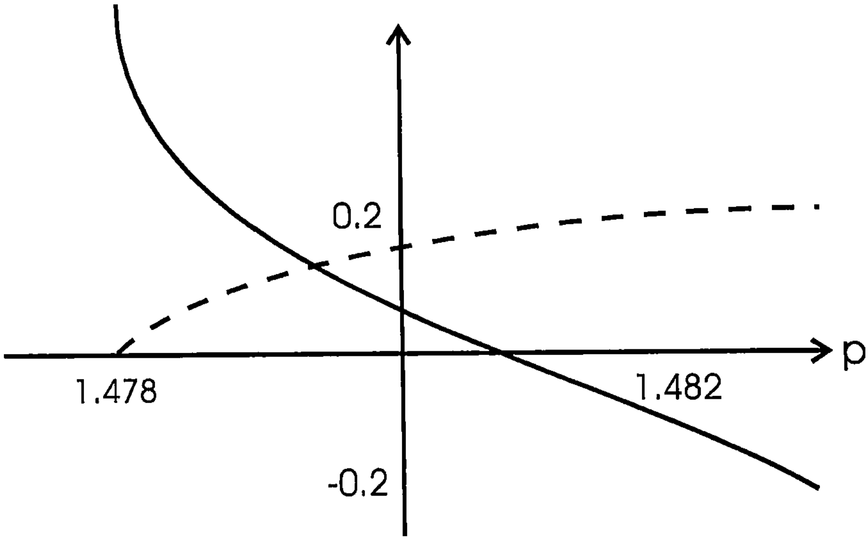

FIGURE 3 Graphic solution of the trascendental Eq. (23) as a function of the propagation constant p for *WR*. (—) left hand side, (- - -) right hand side of (23).

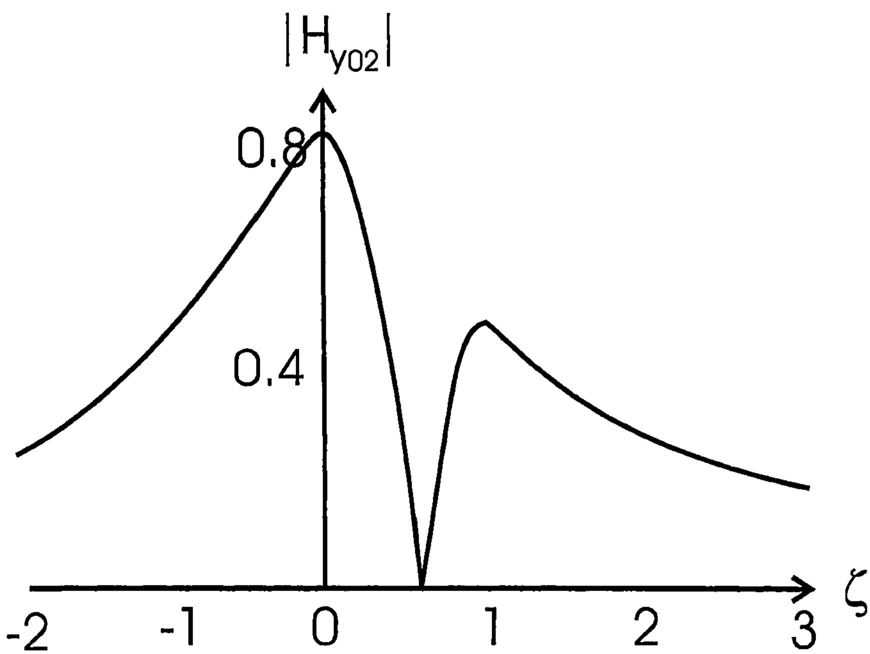

FIGURE 4a Magnitude of field distribution for the mode $|\overline{H}_{y02}|$ in *WR*.

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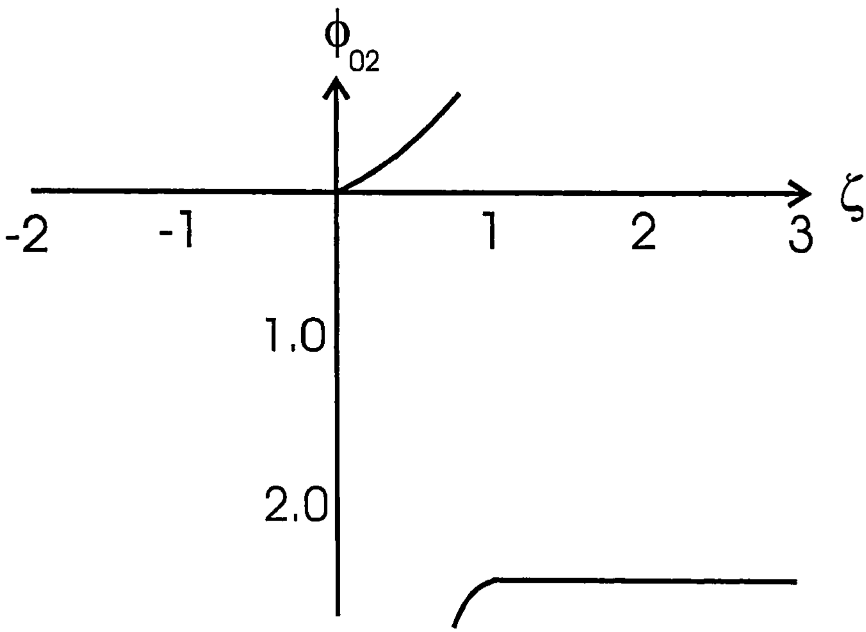

FIGURE 4b Phase distribution ϕ_{02}^{WR} for the mode $|\overline{H}_{y02}|$ in WR.

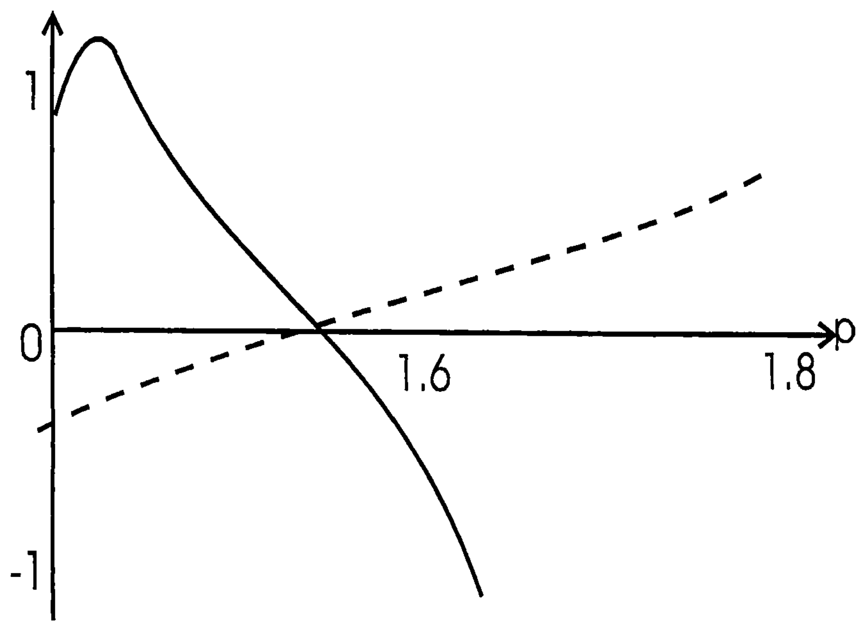

FIGURE 5 The same as in Figure 3 for Eq. (35) for SR.

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plotted in Figure 6a. If we compare the position of the maximum of $|\overline{H}_{y01}|$ with the corresponding one obtained by Lin *et al.* [1], we find a difference of 12%. On the other hand, Figure 6b shows the phase $\phi_{01}^{SR}(\zeta, k_0)$ of $|\overline{H}_{y01}|$, which when compared with the corresponding one obtained by Lin *et al.* [1], we find that their maximum values differ in 7%. This comparison shows that there is a larger disagreement for the strong regime between our analytical calculation and the numerical ones performed by Lin *et al.* [1], but the comparison is still quite good.

It is important to point out that the value of the expansion parameter $k_0l = 6.28$ is not even one order of magnitude larger than unity; so strictly speaking, it is really out of the validity interval for the WKB approximation. In spite of this as mentioned above, the comparison with the numerical results was reasonably good since the difference was only 12%.

Finally, let us address some of the limitations and contributions of our method. Although our analytical method is an approximate one, it allows us to get a better physical insight into the mechanisms of propagation along the slab by means of the optical beam ray trajectories and analytical expressions for the TM modes. It also provides for some additional physical properties

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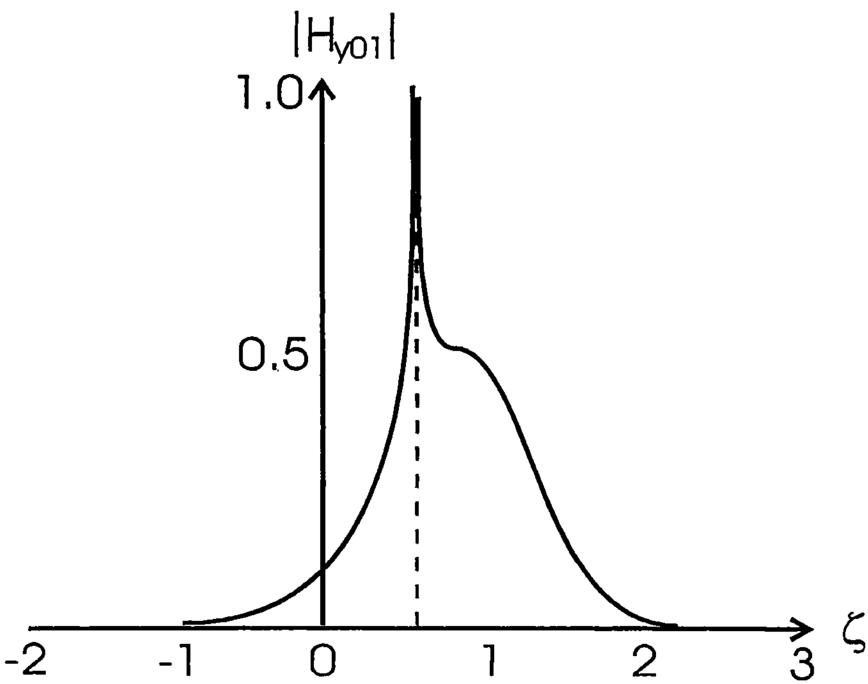

FIGURE 6a Magnitude of field distribution for the mode $|\overline{H}_{y01}|$ in *SR*.

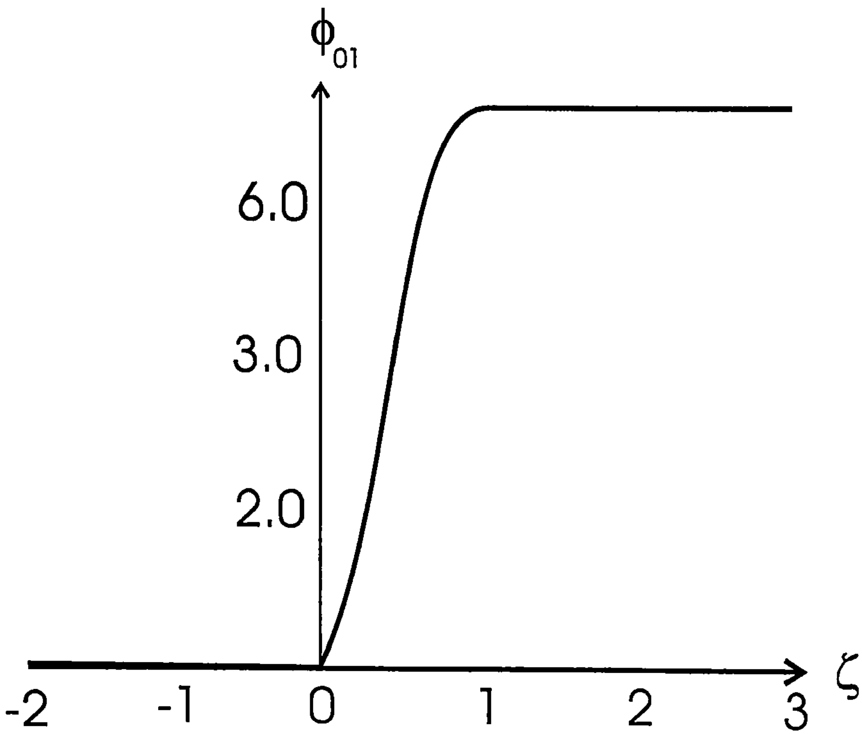

FIGURE 6b Phase distribution ϕ_{01}^{SR} for the mode $|\overline{H}_{y01}|$ in SR.

of the waveguide, such as the waveguide dispersion relation, the cutoff frequency, the maximum number of modes and the spatial distribution of the energy density, quantities which are important in the design of a waveguide.

Although the WKB approximation used here is less exact when the parameter $k_0 l$ is smaller than 10, we still found a good agreement with the exact numerical calculations. However, it would be of interest to extend the exact numerical calculations to higher values of $k_0 l$ to assess if the comparison improves.

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